A pulsating sphere in a rotating fluid

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A pulsating sphere, which performs a sequence of virtually impulsive changes in its radius with time, is completely surrounded by an inviscid, incompressible fluid whose velocity field is generally rotational. This paper indicates how it is possible, by means of Helmholtz's theorem, to relate the corresponding vorticity and velocity fields immediately before and after such expansions or contractions.

The method is then applied to the case of a spherical mass of fluid initially in uniform rotation in which a spherical core undergoes a single sudden expansion, followed after a short interval by an equally rapid contraction back to the original radius. An interesting meridional flow is thereby induced, which tends to decrease the angular velocity of rotation of the fluid near the poles at the outer surface, relative to that of the equatorial fluid. It is perhaps significant that this is in qualitative agreement with the variation of angular velocities observed at the surface of the sun.

1. Introduction

A mass of inviscid, incompressible fluid contains a solid (or possibly fluid) sphere, centred about $\mathbf{r} = 0$, whose radius, $a(t)$, varies with time. The sphere is said to be *pulsating*. Suppose from the outset that these pulsations consist of expansions and/or contractions which occur almost instantaneously, interspersed among intervals during which the radius remains constant. This restriction is hardly a severe one, for any time-history can be approximated by a suitably chosen series of such 'jumps'. The sphere radius can be written as

$$\left.\begin{array}{lll} a = a_0 & \text{when} & t_0 < t < t_1, \\ \cdots & \cdots\ \cdots & \cdots\ \cdots\ \cdots \\ a = a_n & \text{when} & t_n < t < t_{n+1}, \text{etc.} \end{array}\right\} \quad (1)$$

Assume also that the surrounding fluid does not cavitate during the sudden motions.

The velocity field, $\mathbf{v}(\mathbf{r}, t)$, and hence the vorticity field, $\mathbf{w} = \operatorname{curl} \mathbf{v}$, is arbitrarily specified throughout this fluid at some particular time. If the fluid is of infinite extent, it is desirable that $\mathbf{w}$ should approach a constant $\mathbf{w}_\infty$ sufficiently rapidly as $r \to \infty$, so that a vector potential

$$\mathrm{A}(\mathbf{r}) = \int_V \frac{\mathbf{w}(\mathbf{r}') - \mathbf{w}_\infty}{|\mathbf{r}' - \mathbf{r}|} d\mathbf{r}'$$

can exist. If it is bounded, then $\mathbf{n}.\mathbf{v}$ should vanish at the surfaces, except during the sudden changes in the radius of the sphere, when the boundaries would adjust to keep the fluid volume constant.

$\mathbf{v}(\mathbf{r})$ is generally not steady, even in the absence of any motion by the sphere. However, its behaviour during the quiescent intervals can at least in principle be calculated by standard methods, and does not therefore concern us here; only what happens to $\mathbf{v}$ and $\mathbf{w}$ as a result of the quasi-impulsive changes in radius appears to be in question. It will now be shown how the values of $\mathbf{v}$ and $\mathbf{w}$ just following an expansion are related to those that existed immediately before.

2. Effect on the vorticity field of a sudden expansion

Consider a sudden increase in the radius of the sphere from a_0 to a_1 which occurs at time t_1. Take any two infinitesimally separated fluid particles lying outside the sphere, say those at P and Q, which at t_1- are joined by a particular vortex filament Ω (see figure 1). Clearly, the expansion displaces them outwards to positions P' and Q' respectively. PP' and QQ' may be called the 'drifts' of these particles caused by the expansion.

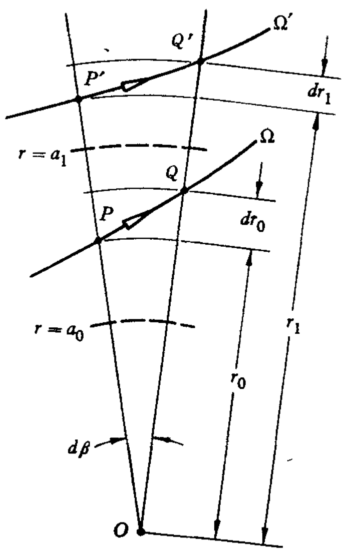

FIGURE 1. Positions of a typical vortex filament immediately before and after an expansion of the sphere.

Since the expansion takes place in virtually an instant, the velocities associated with it must be immensely larger than the original velocities of the fluid. It follows that the 'drifts' are just the same as if the body had instead expanded less rapidly into an initially stationary fluid. The instantaneous displacements of the fluid particles are radial and they will obey the increase-of-volume condition:

$$r_1^3 - r_0^3 = a_1^3 - a_0^3 = A_1^3 = \text{const.} \qquad (2)$$

Helmholtz's theorem states that vortex filaments move with the fluid. There is no reason to suspect an exception in the present case, for it has been assumed that the expansion, though fast, is not quite instantaneous. P' and Q' are therefore on the same vortex line, Ω'. Referring again to figure 1, the sudden expansion of the sphere has thus the following effect on the components of *vorticity*, which are proportional to the infinitesimal distance increments:

The radial component is *decreased* by the ratio $dr_1/dr_0 = r_0^2/r_1^2$; whereas the lateral components are *increased* $r_1 d\beta/r_0 d\beta = r_1/r_0$ times. Or, in spherical polar co-ordinates (r = radius; θ = co-latitude; ϕ = longitude),

$$\mathbf{w}(t_1+) = \{w_r'; w_\theta'; w_\phi'\} = \left\{\frac{r_0^2}{r_1^2}w_r; \frac{r_1}{r_0}w_\theta; \frac{r_1}{r_0}w_\phi\right\}, \tag{3}$$

the unprimed w's being the components of $\mathbf{w}(t_1-)$.

It does not appear possible to relate the before-and-after components of *velocity* quite so directly. Though $\mathbf{v}(t_1+)$ is strictly determinate from $\mathbf{w}(t_1+)$ and the boundary conditions,† it will generally have to be calculated by means of the vector potential. There is, however, one case where the transformation is simplified and that is when the component of velocity tangential to the circles r and θ constant, i.e. v_ϕ, is independent of the angle ϕ. Then it is easy to prove, since such rings at $t = t_1-$ have been shifted into larger circles by the time t_1+, and since the circulation theorem is obeyed, that

$$\mathbf{v}(t_1+) = \left\{v_r'; v_\theta'; v_\phi' = \frac{r_0}{r_1}v_\phi\right\}. \tag{4}$$

Equations (3) and (4) apply of course without change to a contraction by the sphere; only then $r_1 < r_0$.

3. Uniformly rotating fluid

Let us now apply the considerations of the previous section to a relatively simple example. Take the original flow to be one of uniform rotation, that is,

$$\mathbf{v}(t < t_1) = \tfrac{1}{2}\mathbf{w}_0 \times \mathbf{r}_0 = \{0; 0; \tfrac{1}{2}w_0 r_0 \sin\theta\}, \tag{5}$$

the vorticity for $t < t_1$ is

$$\mathbf{w}(t < t_1) = \{w_0 \cos\theta; -w_0 \sin\theta; 0\}, \tag{6}$$

from which equation (3) gives immediately

$$\mathbf{w}(t_1+) = \{w_0(r_0/r_1)^2 \cos\theta; -w_0(r_1/r_0)\sin\theta; 0\}. \tag{7}$$

This happens to be the kind of special case in which equation (4) applies; thus we know v_ϕ' immediately. Now it is quickly seen that v_ϕ' itself already possesses the vorticity given by equation (7) required of the velocity field. Hence, as long as the normal velocity at the sphere surface is zero, it follows that

$$\mathbf{v}(t_1+) = \{0; 0; \tfrac{1}{2}w_0 r_1 (r_0/r_1)^2 \sin\theta\}. \tag{8}$$

Let us examine the effect of these velocities upon the vortex filaments over a short interval of time, Δt, after t_1. For the fluid to continue rotating rigidly, v_ϕ would have to equal $v_{\phi\,\mathrm{rigid}} = \tfrac{1}{2}w_0 r_1 \sin\theta$. But this is not the case here, for $v_\phi < v_{\phi\,\mathrm{rigid}}$. In fact, all particles, particularly those in the vicinity of the sphere, will increasingly 'lag' behind their would-be positions of uniform rotation as time goes on, and the vortex lines consequently suffer a certain distortion. This is illustrated in figure 2.

† If the fluid extends to infinity, the boundary condition $\mathbf{v}(\mathbf{r}_0; t_1-) = \mathbf{v}(\mathbf{r}_1; t_1+)$ applies there.

Note that since $\mathbf{v}.\mathbf{w} = 0$ at t_1+ the motion cannot in the first instance stretch the vortex lines in the direction of $\mathbf{w}(t_1+)$. Rather, any stretching that does occur gives rise to a lateral component of vorticity, $\Delta w'_\phi$. At $t_1+\Delta t$, one can therefore separate the vorticity field into a *primary field*, $\mathbf{w}^*$, which in the first approximation is unchanged from $\mathbf{w}$ at $t = t_1+$, and a *secondary or induced field*, $\Delta\mathbf{w}'$, composed of vortex rings that are concentric with the axis of symmetry. The

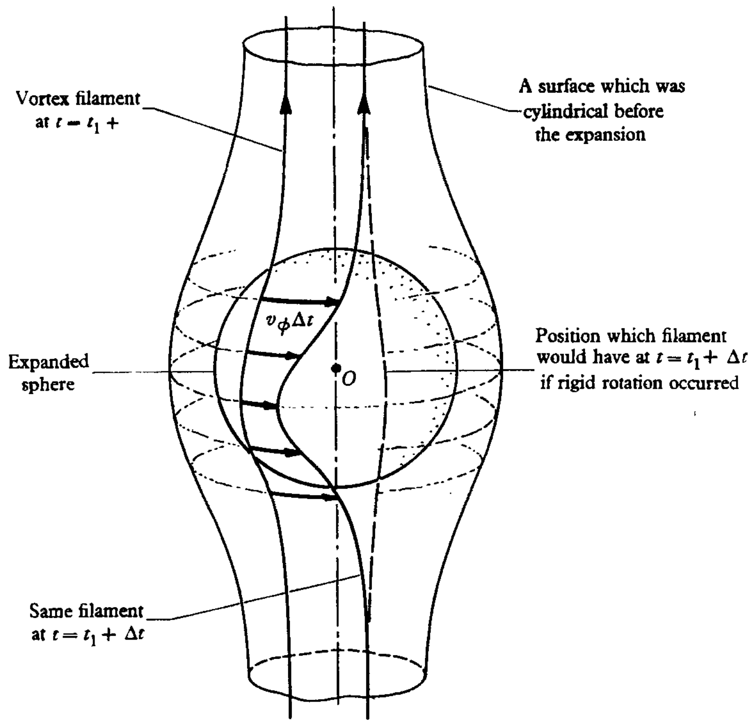

FIGURE 2. The distortion of a typical vortex filament during a short interval after an expansion.

intensity of the latter grows very nearly linearly with time, and it is antisymmetric about the equatorial plane. These rings of vorticity represent a meridional flow directed inwards near the equatorial plane, and outwards near the axis.

Suppose now, further, that at $t = t_1+\Delta t = t_2$ the sphere undergoes a rapid contraction back to the original radius a_0. Reversing the previous procedure, we see that the primary vorticity field transforms back to the constant $\mathbf{w}_0$ which existed before t_1. However, the contraction cannot cause the vanishing of the secondary vorticity, which sprang up during the interval Δt. In other words, the expansion-contraction cycle of the sphere (i.e. the single pulsation) leaves as an immediate after-effect a vortex field $\Delta\mathbf{w}''(r_0, \theta) = (r_0/r_1)\,\Delta\mathbf{w}'(r_1, \theta)$, together with its associated flow in the originally uniformly rotating fluid, and the motion is not steady any more.

This secondary vorticity field, $\Delta\mathbf{w}''$, can be calculated from the distortion of the vortex lines at t_2+. If g represents the 'lag' of fluid particles at that time behind the positions which they would have reached in the absence of the pulsation, then

$$\Delta w''_\phi = -w_0 \frac{\partial g}{\partial z}, \tag{9}$$

z being measured along the axis of rotation. Now g equals (r_0/r_1) times Δt multiplied by the velocity deficiency during the expanded phase, which is $\frac{1}{2}w_0 r_1[1-(r_0/r_1)^2]\sin\theta$. Differentiating, one finds that

$$\Delta w''_\phi = \tfrac{1}{2}A_1^3 w_0^2 \Delta t (r_0^2/r_1^5)\sin 2\theta, \tag{10}$$

where $r_1^3 = r_0^3 + A_1^3$.

In trying to find velocities to match $\Delta w''_\phi$, assume that $r_0^3 \gg A_1^3$. This means that the solutions are not going to be accurate for small radii, but, fortunately, this is unimportant. In this case equation (10) can be written in the expanded form

$$\Delta w''_\phi \simeq A_1^3 w_0^2 \Delta t\, r_0^{-3} \sin 2\theta\,[1-\tfrac{5}{3}(A_1/r_0)^3+\ldots]. \tag{11}$$

It can then be established that

$$q_r(t_2+) \simeq \tfrac{1}{2}A_1^3 w_0^2 \Delta t\, r_0^{-2}[1+\tfrac{5}{3}(A_1/r_0)^3]\,[\cos^2\theta-\tfrac{1}{3}] \tag{12a}$$

and

$$q_\theta(t_2+) \simeq \tfrac{5}{12}A_1^6 w_0^2 \Delta t\, r_0^{-5}\sin 2\theta \tag{12b}$$

are divergence-free and consistent with (11). Together with a potential-derived velocity field, which will be chosen to satisfy the boundary conditions (if any), (12*a*) and (12*b*) describe the induced flow immediately after a single pulsation.

4. Sphere of rotating fluid with a pulsating core

An interesting situation arises when the pulsating sphere is at the centre of a rotating mass of fluid which itself is spherical. In this case, there are two boundary conditions at t_2+. The normal velocity must vanish at the outer boundary, where $r = b$, say, and perhaps at the core where $r = a_0$. However, it is reasonable to disregard the latter, for the resulting irrotational velocities would drop off with distance as rapidly as r_0^{-4}, and if the core were fluid, such a condition would be unrealistic anyway. To be accurate, we assume that $a_0^3 \ll b^3$.

The boundary condition at $r = b$ is satisfied by our superimposing upon the q's the velocities derived from the potential $\phi = Kr_0^2[\cos^2\theta-\frac{1}{3}]$. The constant K is determined from $-(-\partial\phi/\partial r_0)_{r_0=b} = q_r(b)$, the latter being given by equation (12*a*). The resulting complete induced flow in the meridional planes immediately after the re-contraction, except near the centre, is very nearly

$$v_r(t_2+) \simeq \tfrac{1}{2}A_1^3 w_0^2 \Delta t\, r_0^{-2}[1-(r_0/b)^3]\,[\cos^2\theta-\tfrac{1}{3}] \tag{13a}$$

and

$$v_\theta(t_2+) \simeq \tfrac{1}{4}A_1^3 w_0^2 \Delta t\, r_0 b^{-3}\sin 2\theta. \tag{13b}$$

Higher-order terms have been omitted from these equations.

Now it must be emphasized that these velocities are transient, for they will gradually distort the primary vortex filaments with resulting complications to the flow, which we cannot consider here. Moreover, in a real fluid viscosity will at the same time tend to destroy meridional motions and to restore the entire sphere to uniform rotation (assuming that the surface at $r = b$ is frictionless) with the initial angular velocity.

But neither of these effects is likely to contradict our conclusion that one of the effects of an expansion-contraction of the sphere within the sphere of rotating fluid is a flow along the surface from both poles towards the equator. In fact, we might go one step further and postulate that the core pulsates periodically, expanding and then contracting, perhaps with sufficient quiescent intervals between a contraction and the next expansion to permit what viscosity there may exist to restore the rotation to nearly uniform. In this case, a one-way surface flow away from the poles will be present throughout, though it will fluctuate in intensity.

It is possible to show that such a meridional flow along the surface cannot help but exercise a perhaps disproportionate effect upon the main peripheral velocities there, namely, the v_ϕ's. Consider any ring of constant latitude fluid on this sphere at some given time. It has a certain circulation, say Γ. If we now assume negligible viscosity, as the meridional velocities displace this ring of fluid towards the equator and its radius increases, Γ remains constant owing to the circulation theorem. Therefore the peripheral velocities, v_ϕ, along it are decreased. By such arguments, if carried to the limit, it would follow that the fluid at the poles would cease revolving altogether! In reality, the situation will not be so drastic, but clearly if we start with a sphere rotating uniformly with the angular velocity $\frac{1}{2}w_0$ and superimpose meridional velocities such as we have described upon it, the angular velocity at the poles will soon be less than $\frac{1}{2}w_0$.

At the low latitudes, however, fluid rings increase less in radius, and because particles on the surface must remain there, a similar reasoning would reveal that the rotational speed is unaltered at the equator. One would expect the combined trend to be held in check, both by higher-order effects here neglected, and by the action of viscosity. Nevertheless, qualitatively, the regular pulsation of the core will have an effect whereby fluid at increasing latitudes will have a progressively smaller *angular* velocity, and thus take longer to complete a rotation.

This remarkable effect has been observed on the sun. Whereas its equatorial (sidereal) period of rotation has been measured at about 25 days, this period is known to grow with latitude, reaching about 29 days at the 60° latitude. This phenomenon seems to have remained virtually unexplained.

There is one aspect of a typical pulsation which has been taken rather for granted so far. We have really only considered pulsations where the core expands and then contracts, followed and preceded by periods of quiescence. It should be noted that if the behaviour were the reverse—that is, the core first contracting, then expanding—our deductions, too, would be exactly the opposite, and in the case of the spherical mass of fluid surface currents would be directed towards the poles. That its attendant effects are not evidenced in the sun's rotation suggests, according to this theory, that the hypothetical pulsations of its core are of an explosive, and not an implosive, nature.

An apologia is perhaps due for citing as a possible example the sun, which hardly consists of incompressible fluid. However, the reasons for restricting direct analysis to such fluids are obvious; and so are those for idealizing the pulsations as quasi-impulsive. It is reasonable to suppose that the qualitative conclusions are valid no matter whether the fluid is incompressible or not. Finally, we may observe that expanding spherical shock-waves have an effect somewhat analogous to the expansions–contractions of a core; both displace fluid temporarily outwards to an extent varying with distance from the centre.